

BOUNDEDNESS IN PERTURBED NONLINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

SANG IL CHOI* AND YOON HOE GOO**

ABSTRACT. In this paper, we investigate bounds for solutions of the perturbed nonlinear functional differential systems with a t_∞ -similarity condition using the notion of h -stability.

1. Introduction and preliminaries

We consider the nonlinear nonautonomous differential system

$$(1.1) \quad x'(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and \mathbb{R}^n is the Euclidean n -space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed nonlinear functional differential systems of (1.1)

$$(1.2) \quad y' = f(t, y) + \int_{t_0}^t g(s, y(s)) ds + h(t, y(t), Ty(t)), \quad y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0) = 0$, $h(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \rightarrow C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator. For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. For an $n \times n$ matrix A , define the norm $|A|$ of A by $|A| = \sup_{|x| \leq 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then, we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$(1.3) \quad v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0$$

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Correspondence should be addressed to Yoon Hoe Goo, yhgoo@hanseo.ac.kr.

and

$$(1.4) \quad z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0.$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.3).

We recall some notions of h -stability [15].

DEFINITION 1.1. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called an h -system if there exist a constant $c \geq 1$, and a positive continuous function h on \mathbb{R}^+ such that

$$|x(t)| \leq c |x_0| h(t) h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

DEFINITION 1.2. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called

(hS) h -stable if there exists $\delta > 0$ such that (1.1) is an h -system for $|x_0| \leq \delta$ and h is bounded.

The notion of h -stability (hS) was introduced by Pinto [15,16] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. He obtained a general variational h -stability and some properties about asymptotic behavior of solutions of differential systems called h -systems. Choi, Ryu [3] and Choi, Koo, and Ryu [4] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [7,8,9] and Goo et al. [11] investigated boundedness of solutions for nonlinear perturbed systems.

In this paper, we investigate bounds for solutions of the perturbed nonlinear functional differential systems using the notion of t_∞ -similarity.

Let \mathcal{M} denote the set of all $n \times n$ continuous matrices $A(t)$ defined on \mathbb{R}^+ and \mathcal{N} be the subset of \mathcal{M} consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞ -similarity in \mathcal{M} was introduced by Conti [6].

DEFINITION 1.3. A matrix $A(t) \in \mathcal{M}$ is t_∞ -similar to a matrix $B(t) \in \mathcal{M}$ if there exists an absolutely integrable $n \times n$ matrix $F(t)$ over \mathbb{R}^+ , i.e.,

$$\int_0^\infty |F(t)| dt < \infty$$

such that

$$(1.5) \quad \dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t)$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞ -similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on \mathbb{R}^+ , and it preserves some stability concepts [6, 12].

We give some related properties that we need in the sequel.

LEMMA 1.4. [16] *The linear system*

$$(1.6) \quad x' = A(t)x, \quad x(t_0) = x_0,$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h -system (respectively h -stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on \mathbb{R}^+ such that

$$(1.7) \quad |\phi(t, t_0)| \leq c h(t) h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (1.6).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$(1.8) \quad y' = f(t, y) + g(t, y), \quad y(t_0) = y_0,$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 1.5. *If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s)) g(s, y(s)) ds.$$

THEOREM 1.6. [3] *If the zero solution of (1.1) is hS , then the zero solution of (1.3) is hS .*

THEOREM 1.7. [4] *Suppose that $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (1.3) is hS , then the solution $z = 0$ of (1.4) is hS .*

LEMMA 1.8. (*Bihari – type inequality*) *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u . Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda(s)w(u(s))ds, \quad t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \lambda(s) ds \right], \quad t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^u \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \lambda(s) ds \in \text{dom} W^{-1} \right\}.$$

LEMMA 1.9. [5] Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u . Suppose that for some $c > 0$,

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) w(u(s)) ds + \int_{t_0}^t \lambda_2(s) \left(\int_{t_0}^s \lambda_3(\tau) w(u(\tau)) d\tau \right) ds, \quad 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau) ds) \right], \quad t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau) d\tau) ds \in \text{dom} W^{-1} \right\}.$$

LEMMA 1.10. [9] Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s) u(s) ds + \int_{t_0}^t \lambda_2(s) w(u(s)) ds + \int_{t_0}^t \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) u(\tau) d\tau ds.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau) ds \right], \quad t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau) ds \in \text{dom} W^{-1} \right\}.$$

2. Main results

In this section, we investigate boundedness for solutions of the non-linear perturbed differential systems via t_∞ -similarity.

To obtain the bounded result, the following assumptions are needed:

(H1) $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$.

(H2) The solution $x = 0$ of (1.1) is hS with the increasing function h .

THEOREM 2.1. Let $a, b, c, q, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $u \leq w(u)$ and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that (H1), (H2), and g in (1.2) satisfies

$$(2.1) \quad \int_{t_0}^t |g(s, y(s))| ds \leq a(t)|y(t)|, \\ |h(t, y(t), Ty(t))| \leq b(t)w(|y(t)|) + c(t)|Ty(t)|,$$

and

$$(2.2) \quad |Ty(t)| \leq \int_{t_0}^t q(s)|y(s)| ds,$$

where $a, b, c, q \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (a(s) + c(s) + b(s)) \int_{t_0}^s k(\tau) d\tau + c(s) \int_{t_0}^s q(\tau) d\tau ds \right],$$

$t_0 \leq t < b_1$, where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t (a(s) + c(s) + b(s)) \int_{t_0}^s k(\tau) d\tau \right. \\ \left. + c(s) \int_{t_0}^s q(\tau) d\tau ds \in \text{dom}W^{-1} \right\}.$$

Proof. Using the nonlinear variation of constants formula of Alekseev [1], any solution $y(t) = y(t, t_0, y_0)$ of (1.2) passing through (t_0, y_0) is given by

$$(2.3) \quad y(t, t_0, y_0) = x(t, t_0, y_0) \\ + \int_{t_0}^t \Phi(t, s, y(s)) \left(\int_{t_0}^s g(\tau, y(\tau)) d\tau + h(s, y(s), Ty(s)) \right) ds.$$

By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. In view of Lemma 1.4, the hS condition of $x = 0$ of (1.1), (2.1), (2.2), and (2.3), we have

$$\begin{aligned}
|y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left(\int_{t_0}^s |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))| \right) ds \\
&\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) h(s)^{-1} \left(a(s) |y(s)| + b(s) w(|y(s)|) \right. \\
&\quad \left. + c(s) \int_{t_0}^s q(\tau) |y(\tau)| d\tau \right) ds \\
&\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) (a(s) \frac{|y(s)|}{h(s)} \\
&\quad + b(s) w(\frac{|y(s)|}{h(s)}) + c(s) \int_{t_0}^s q(\tau) \frac{|y(\tau)|}{h(\tau)} d\tau) ds.
\end{aligned}$$

Set $u(t) = |y(t)| |h(t)|^{-1}$. Then, an application of Lemma 1.10 yields

$$|y(t)| \leq h(t) W^{-1} \left[W(c) + c_2 \int_{t_0}^t (a(s) + b(s) + c(s) \int_{t_0}^s q(\tau) d\tau) ds \right]$$

where $c = c_1 |y_0| h(t_0)^{-1}$. Hence, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. \square

REMARK 2.2. Letting $w(u) = u$ and $b(t) = c(t) = 0$ in Theorem 2.1, we obtain the same result as that of Theorem 3.3 in [10].

LEMMA 2.3. Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$,

$$\begin{aligned}
(2.4) \quad u(t) &\leq c + \int_{t_0}^t \lambda_1(s) u(s) ds + \int_{t_0}^t \lambda_2(s) w(u(s)) ds + \int_{t_0}^t \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) u(\tau) d\tau ds \\
&\quad + \int_{t_0}^t \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) w(u(\tau)) d\tau ds, \quad 0 \leq t_0 \leq t.
\end{aligned}$$

Then

$$\begin{aligned}
(2.5) \quad u(t) &\leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau \right. \\
&\quad \left. + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) d\tau) ds \right],
\end{aligned}$$

$t_0 \leq t < b_1$, where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) d\tau) ds \in \text{dom}W^{-1} \right\}.$$

Proof. Define a function $z(t)$ by the right member of (2.4). Then, we have $z(t_0) = c$ and

$$\begin{aligned} z'(t) &= \lambda_1(t)u(t) + \lambda_2(t)w(u(t)) + \lambda_3(t) \int_{t_0}^t \lambda_4(s)u(s)ds \\ &\quad + \lambda_5(t) \int_{t_0}^t \lambda_6(s)w(u(s))ds \\ &\leq (\lambda_1(t) + \lambda_2(t) + \lambda_3(t) \int_{t_0}^t \lambda_4(s)ds \lambda_5(t) \int_{t_0}^t \lambda_6(s)ds) w(z(t)), \quad t \geq t_0, \end{aligned}$$

since $z(t)$ and $w(u)$ are nondecreasing, $u \leq w(u)$, and $u(t) \leq z(t)$. Therefore, by integrating on $[t_0, t]$, the function z satisfies

$$(2.6) \quad \begin{aligned} z(t) &\leq c + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^s \lambda_4(\tau) d\tau) w(z(s)) \\ &\quad + \lambda_5(s) \int_{t_0}^s \lambda_6(\tau) d\tau) w(z(s)) ds. \end{aligned}$$

It follows from Lemma 1.8 that (2.6) yields the estimate (2.5). □

THEOREM 2.4. *Let $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $u \leq w(u)$ and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that (H1), (H2), and g in (1.2) satisfies*

$$(2.7) \quad \int_{t_0}^t |g(s, y(s))| ds \leq a(t)w(|y(t)|) + b(t) \int_{t_0}^t k(s)|y(s)| ds, \quad t \geq t_0 \geq 0$$

and

$$(2.8) \quad |h(t, y(t), Ty(t))| \leq c(t)(|y(t)| + |Ty(t)|), \quad |Ty(t)| \leq \int_{t_0}^t q(s)w(|y(s)|) ds,$$

$t \geq t_0 \geq 0$, where $a, b, c, k, q \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (a(s) + c(s) + b(s) \int_{t_0}^s k(\tau) d\tau + c(s) \int_{t_0}^s q(\tau) d\tau) ds \right],$$

$t_0 \leq t < b_1$, where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t (a(s) + c(s) + b(s) \int_{t_0}^s k(\tau) d\tau + c(s) \int_{t_0}^s q(\tau) d\tau) ds \in \text{dom} W^{-1} \right\}.$$

Proof. It is known that the solution of (1.2) is represented by the integral equation (2.3). By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. By Lemma 1.4, the hS condition of $x = 0$ of (1.1), (2.3), (2.7), and (2.8), we have

$$\begin{aligned} |y(t)| &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) h(s)^{-1} ((a(s)w(|y(s)|) \\ &\quad + b(s) \int_{t_0}^s k(\tau) |y(\tau)| d\tau + c(s)(|y(s)| + \int_{t_0}^s q(\tau)w(|y(\tau)|) d\tau) ds \\ &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) (c(s) \frac{|y(s)|}{h(s)} + a(s)w(\frac{|y(s)|}{h(s)})) ds \\ &\quad + \int_{t_0}^t c_2 h(t) (b(s) \int_{t_0}^s k(\tau) \frac{|y(\tau)|}{h(\tau)} d\tau + c(s) \int_{t_0}^s q(\tau)w(\frac{|y(\tau)|}{h(\tau)}) d\tau) ds. \end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, an application of Lemma 2.3 yields

$$\begin{aligned} |y(t)| &\leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (a(s) + c(s) + b(s) \int_{t_0}^s k(\tau) d\tau \right. \\ &\quad \left. + c(s) \int_{t_0}^s q(\tau) d\tau) ds \right], \end{aligned}$$

where $c = c_1 |y_0| h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$. Hence, the proof is complete. \square

REMARK 2.5. Letting $c(t) = 0$ in Theorem 2.4, we obtain the same result as that of Theorem 3.4 in [7].

THEOREM 2.6. Let $a, b, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that (H1), (H2), and g in (1.2) satisfies

$$(2.9) \quad |g(t, y(t))| \leq a(t)w(|y(t)|), |h(t, y(t), Ty(t))| \leq b(t)w(|y(t)|),$$

where $a, b \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (b(s) + \int_{t_0}^s a(\tau)d\tau)ds \right],$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t (b(s) + \int_{t_0}^s a(\tau)d\tau)ds \in \text{dom}W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Applying Lemma 1.4, the hS condition of $x = 0$ of (1.1), (2.3) and (2.9), we have

$$\begin{aligned} |y(t)| &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t)h(s)^{-1} \left(\int_{t_0}^s a(\tau)w(|y(\tau)|)d\tau \right. \\ &\qquad \qquad \qquad \left. + b(s)w(|y(s)|) \right) ds \\ &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t) \left(b(s)w\left(\frac{|y(s)|}{h(s)}\right) + \int_{t_0}^s a(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right)d\tau \right) ds. \end{aligned}$$

Defining $u(t) = |y(t)||h(t)|^{-1}$, then, by Lemma 1.9, we have

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (b(s) + \int_{t_0}^s a(\tau)d\tau)ds \right],$$

where $c = c_1|y_0|h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$. This completes the proof. \square

REMARK 2.7. Letting $b(t) = 0$ in Theorem 2.6, we obtain the similar result as that of Theorem 3.5 in [11].

LEMMA 2.8. Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,

$$\begin{aligned} (2.10) \quad u(t) &\leq c + \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_2(s) \int_{t_0}^s \left(\lambda_3(\tau)w(u(\tau)) \right. \\ &\qquad \qquad \qquad \left. + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)u(r)dr \right) d\tau ds. \end{aligned}$$

Then

$$(2.11) \quad u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) dr) d\tau) ds \right],$$

$t_0 \leq t < b_1$, where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) dr) d\tau) ds \in \text{dom} W^{-1} \right\}.$$

Proof. Define a function $z(t)$ by the right member of (2.10). Then, we have $z(t_0) = c$ and

$$\begin{aligned} z'(t) &= \lambda_1(t)u(t) + \lambda_2(t) \int_{t_0}^t (\lambda_3(s)w(u(s)) + \lambda_4(s) \int_{t_0}^s \lambda_5(\tau)u(\tau) d\tau) ds \\ &\leq (\lambda_1(t) + \lambda_2(t) \int_{t_0}^t (\lambda_3(s) + \lambda_4(s) \int_{t_0}^s \lambda_5(\tau) d\tau) ds) w(z(t)), \quad t \geq t_0, \end{aligned}$$

since $z(t)$ and $w(u)$ are nondecreasing, $u \leq w(u)$, and $u(t) \leq z(t)$. Therefore, by integrating on $[t_0, t]$, the function z satisfies

$$(2.12) \quad z(t) \leq c + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) dr) d\tau) w(z(s)) ds.$$

It follows from Lemma 1.8 that (2.12) yields the estimate (2.11). \square

THEOREM 2.9. Let $a, b, c, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $u \leq w(u)$ and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that (H1), (H2), and g in (1.2) satisfies

$$(2.13) \quad |g(t, y(t))| \leq a(t)w(|y(t)|) + b(t) \int_{t_0}^t k(s)|y(s)| ds$$

and

$$(2.14) \quad |h(t, y(t), Ty(t))| \leq c(t)|y(t)|,$$

where $a, b, c, k \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t (c(s) + \int_{t_0}^s (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r) dr) d\tau) ds \right],$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t (c(s) + \int_{t_0}^s (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r) dr) d\tau) ds \in \text{dom} W^{-1} \right\}.$$

Proof. It is well known that the solution of (1.2) is represented by the integral equation (2.3). By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula (2.3), Lemma 1.4, the hS condition of $x = 0$ of (1.1), (2.13), and (2.14), we have

$$\begin{aligned} |y(t)| &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) h(s)^{-1} \left(\int_{t_0}^s (a(\tau) w(|y(\tau)|) \right. \\ &\quad \left. + b(\tau) \int_{t_0}^{\tau} k(r) |y(r)| dr) d\tau + c(s) |y(s)| \right) ds \\ &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) \left(c(s) \frac{|y(s)|}{h(s)} \right. \\ &\quad \left. + \int_{t_0}^s (a(\tau) w\left(\frac{|y(\tau)|}{h(\tau)}\right) + b(\tau) \int_{t_0}^{\tau} k(r) \frac{|y(r)|}{h(r)} dr) d\tau \right) ds. \end{aligned}$$

Set $u(t) = |y(t)| |h(t)|^{-1}$. Then, by Lemma 2.8, we have

$$|y(t)| \leq h(t) W^{-1} \left[W(c) + c_2 \int_{t_0}^t (c(s) + \int_{t_0}^s (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r) dr) d\tau) ds \right],$$

where $c = c_1 |y_0| h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved. \square

REMARK 2.10. Letting $c(t) = 0$ in Theorem 2.9, we obtain the similar result as that of Theorem 3.7 in [7].

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Department of Mathematics
Hanseo University
Seosan 356-706, Republic of Korea
E-mail: schoi@hanseo.ac.kr

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Department of Mathematics
Hanseo University
Seosan 356-706, Republic of Korea
E-mail: yhgoo@hanseo.ac.kr